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Anomalous Fermions

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Based on the recent model of quantum mechanical black holes, it is shown that at energies corresponding to length scales large compared to the Compton wavelength, fermions would exhibit a bosonization in character. It is also argued that in two and one spatial dimensions, fermions would exhibit, in addition, handedness and other features, characteristics which are otherwise suggested by conventional arguments. Finally, all these conclusions are verified and recent experimental confirmation is also cited.

KEY WORDS: Fermi temperature; anomaly.

1. INTRODUCTION

According to a recent model,[(1–5)] the most elementary Fermion, the electron can be treated as a Kerr–Newman type Black Hole bounded by the Compton wavelength, what may be called a Quantum Mechanical Black Hole (QMBH). There is a naked singularity, that is the radius becomes complex, but this is explained by the fact that inside the Compton wavelength there are negative energies manifesting themselves in the form of Zitterbewegung. Indeed the position in the Quantum Mechanical case also becomes complex or equivalently the position operator is non Hermitian. The well known explanation for this is[(6)] that strictly speaking space time points are meaningless, while it is only space time intervals which are meaningful. In Quantum Mechanics as is well known (cf. ref. 6) on averaging over such intervals the non Hermitian Position Operator with complex Eigen values goes over to a Hermitian Operator with real Eigen values. In any case, as is well known, the Kerr–Newman metric describes the field of an electron

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including, and this is remarkable, the Quantum Mechanical anomalous gyro magnetic ratio $g = 2$.

Such a model explains several interesting features like the discreteness of the charge, the lefthandedness of the neutrino and so on. It also leads to a cosmology consistent with observation including a theoretical deduction of the mass, radius and age of the universe and other cosmological parameters and the supposedly mysterious large number coincidences (cf. refs. 1 and 7). Interestingly the cosmological model predicts an ever expanding and accelerating universe, which has been recently observationally confirmed.[8, 9]

On the other hand the model also gives a rationale for weak interactions[10] and for the structure of particles like Baryons and Mesons (cf. ref. 2 and 11).

We now see how from the above model one can argue that at low temperatures Fermions exhibit an anomalous character, indeed as seen in the superfluidity of He^3.[12] We will also briefly comment on Fermionic behaviour in two and one (spatial) dimensions. These considerations are corroborated by conventional theory.

2. ANOMALOUS FERMIONS

According to the above model (cf. ref. 2), the spinorial behaviour of isolated Fermions is a manifestation of the negative energy components which as is known are encountered near the Compton wavelength region.[13] However if we are at scales much greater than the Compton wavelength, that is at low energies, we encounter only positive energy solutions and hence the particle should show up with a Bosonic character.

This conclusion can be immediately verified by the fact that as $v/c \to 0$, or equivalently negative energy components do not contribute (cf. ref. 13), the Dirac equation of the particle with velocity v goes over to the Schrödinger equation in the absence of interactions.

This Bosonization effect is also suggested by the following argument:

For a collection of Fermions, we know that the Fermi energy is given by,[14]

$$\varepsilon_F = p_F^2/2m = \left(\frac{\hbar^2}{2m}\right)\left(\frac{6\pi^2}{v}\right)^{2/3} \tag{1}$$

where $v^{1/3}$ is the interparticle distance. On the other hand, in a different context, for phonons, the maximum frequency is given by, (cf. ref. 14),

$$\omega_m = c\left(\frac{6\pi^2}{v}\right)^{1/3} \tag{2}$$

This occurs for the phononic wavelength $\lambda_m \approx$ inter-atomic distance between the atoms, $v^{1/3}$ being, again, the mean distance between the phonons. "c" in (2) is the velocity of the wave, the velocity of sound in this case. The wavelength λ_m is given by,

$$\lambda_m = \frac{2\pi c}{\omega_m}$$

We can now define the momentum p_m via the de Broglie relation,

$$\lambda_m = \frac{h}{p_m}$$

which gives,

$$p_m = \frac{\hbar}{c}\,\omega_m, \qquad \hbar \equiv \frac{h}{2\pi} \tag{3}$$

We can next get the maximum energy corresponding to the maximum frequency ω_m given by (2),

$$\varepsilon_m = \frac{p_m^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{6\pi^2}{v}\right)^{2/3} \tag{4}$$

Comparing (1) and (4), we can see that ε_m and p_m exactly correspond to ε_F and p_F.

The Fermi energy in (1) is obtained as is known by counting all energy levels below the Fermi energy ε_F using Fermi–Dirac statistics, while the maximum energy in (4) is obtained by counting all energy levels below the maximum value, but by using Bose–Einstein statistics (cf. ref. 14).

We can see why inspite of this, the same result is obtained in both cases. In the case of the Fermi energy, all the lowest energy levels below ε_F are occupied with the Fermionic occupation number $\langle n_p \rangle = 1$, $p < p_F$. Then, the number of levels in a small volume about p is d^3p. This is exactly so for the Bosonic levels also. With the correspondence given in (3), the number of states in both cases coincide and it is not surprising that (1) and (4) are the same.

In effect, Fermions below the Fermi energy should have a strong resemblance to phonons, reminiscent of semions which behave like particles with statistics in between the Fermi–Dirac and Bose–Einstein statistics.[15]

A rationale for the above is the fact that, for $p^2/2m < \varepsilon_F$, as $\Delta p \approx 0$, the levels are very closely spaced and the density of levels in p-space is d^3p as in the case of photons which are Bosons. Bloch's original analysis (cf. ref. 16) corroborates the above considerations.

In any case, for example the conduction electrons in metals can be in the Sommerfeld model, considered to be non interacting Fermions in a box, as also in the Landau theory of Fermi liquids.(17) Moreover the original Tomonaga theory in one dimension which considers the Fermionic ensemble as an ensemble of Bosons and weakly or non interacting Fermions, as in the independent particle model has been found to be true in three dimensions also.(16) It is in this context that we can use the above result that the Dirac equation goes over to the Schrödinger equation at low velocities, to argue that there would be Bosonization effects. So at sufficiently low velocities, that is temperatures, we can expect that these Fermions would exhibit a Bosonic character.

Let us analyse this circumstance further to show that the above conclusions indeed follow from Quantum Field Theory, without any contradiction to the Spin Statistics Theory.

We first show specifically that the Fermi energy corresponds to scales much larger than the Compton wavelength. This follows quite easily. If v is the average volume per particle, then scales much larger than the Compton wavelength imply,

$$v \gg \left(\frac{\hbar}{mc}\right)^3$$

whence, in terms of the Fermi energy, we have (cf. ref. 14)

$$\left(\frac{3\sqrt{\pi}}{4}\right)^{2/3} \cdot \frac{1}{kT\varepsilon_F}\left(\frac{2\pi\hbar^2}{mkT}\right) \gg \left(\frac{\hbar}{mc}\right)^2$$

So,

$$(7.6)\; mc^2 \gg \varepsilon_F$$

Alternatively,

$$\varepsilon_F \sim \frac{1}{N}\sum \frac{p^2}{2m} \ll mc^2, \qquad \text{as} \quad \left(\frac{p}{mc}\right) \ll 1$$

Either way, the length scales associated with the Fermi energy are much greater than the Compton wavelength.

We next observe that the essence of Spin Statistics Theory is that commutators (corresponding to symmetric wave functions) cannot be used with Fermionic fields while anti commutators (corresponding to anti symmetric wave functions), cannot be used with Fermionic fields. However, as is known, this is strictly true at scales not much greater than the Compton wavelength.[18]

Indeed, for a Klein–Gordon field while the vaccum expectation value of the commutator,

$$\langle 0| \, [\phi_r(x), \phi_s(y)] \, |0\rangle \equiv \Delta(x-y)$$

vanishes for space like intervals, the same value for the anti commutator for large spatial distances is given by,

$$\Delta_1'(x, 0) \sim \frac{Ze^{-m\,|x|}}{|x|^2} + \int_{m_1^2}^{\infty} d\sigma^2 \, \rho(\sigma^2) \, \frac{e^{-\sigma\,|x|}}{|x|^2}$$

So this anti commutator is nearly zero for large space like distances, that is the violation of microscopic causality and therefore the Spin Statistics Theory is negligible (cf. ref. 18).

Similarly for Fermionic fields the contradiction arises because, this time the symmetric propagator, the Lorentz Invariant function

$$\Delta(x-x') \equiv \int \frac{d^3k}{(2\pi)^3 \, 3\omega_k} \, [e^{-\imath k \cdot (x-x')} + e^{\imath k \cdot (x-x')}]$$

does not vanish for space like intervals $(x-x')^2 < 0$, where the vacuum expectation value of the commutator is given by the spectral representation,

$$\begin{aligned} S(x-x') &\equiv \imath \, \langle 0| \, [\psi_\alpha(x), \psi_\beta(x')] \, |0\rangle \\ &= -\int dM^2 [\imath \rho_1(M^2) \, \Delta_x + \rho_2(M^2)]_{\alpha\beta} \, \Delta(x-x') \end{aligned}$$

Outside the light cone, $r > |t|$, where $r \equiv |\vec{x} - \vec{x}'|$ and $t \equiv |x_o - x_o'|$, Δ is given by,

$$\Delta(x'-x) = -\frac{1}{2\pi^2 r} \frac{\partial}{\partial r} K_o(m\sqrt{r^2 - t^2})$$

where the modified Bessel function of the second kind, K_o is given by,

$$K_o(mx) = \int_o^{\infty} \frac{\cos(xy)}{\sqrt{m^2 + y^2}} \, dy = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos(xy)}{\sqrt{m^2 + y^2}} \, dy$$

(cf., refs. 19 and 20). In our case, $x \equiv \sqrt{r^2 - t^2}$, and we have,

$$\Delta(x - x') = \text{const}\,\frac{1}{x}\int_{-\infty}^{\infty} \frac{y \sin xy}{\sqrt{m^2 + y^2}}\,dy \sim 0\left(\frac{l}{x}\right)$$

where l is the Compton wavelength ($\hbar = c = 1$).

Once again we can see that the violation with the Spin Statistics Theory is negligible for distances large compared to the Compton wavelength. This confirms the Bosonization effects for low temperature Fermions.

To get yet another alternative justification, we further observe that in the Quantum Field Theory of Fermions, as is well known (cf. ref. 18), the wave function expansion of the Fermion should include solutions of both signs of energy:

$$\psi(\vec{x}, t) = N \int d^3p \sum_{\pm s} [b(p, s)\, u(p, s) \exp(-\imath p^\mu x_\mu/\hbar)$$
$$+ d^*(p, s)\, v(p, s) \exp(+\imath p^\mu x_\mu/\hbar) \tag{5}$$

where N is a normalization constant for ensuring unit probability.

In Quantum Field Theory, the coefficients become creation and annihilation operators while bb^+ and d^+d become the particle number operators with eigen values 1 or 0 only. The Hamiltonian is now given by:[18]

$$H = \sum_{\pm s} \int d^3p\, E_p[b^+(p, s)\, b(p, s) - d(p, s)\, d^+(p, s)] \tag{6}$$

As can be seen from (6), the Hamiltonian is not positive definite and it is this circumstance which necessitates the Fermi–Dirac statistics. In the absence of Fermi–Dirac statistics, the negative energy states are not saturated in the Hole Theory sense so that the ground state would have arbitrarily large negative energy, which is unacceptable. However Fermi–Dirac statistics and the anti commutators implied by it prevent this from happening.

Now in the QMBH model referred to earlier all the negative energies are pinched off inside the Compton wavelength. So, at the scales under consideration, these are inaccessible, so that there is no question of transition to a negative energy level. That is, we do not require Fermi–Dirac statistics, which was invoked only to forbid such a transition. In effect we could work with commutators. This is reminiscent of the Bosonization of Fermions encountered in the one dimensional case (cf. ref. 16).

However it must be observed that the above anomalous behaviour does not mean that the gas behaves classically or that the Pauli Exclusive Principle is inoperative. If that were the case the internal energy at the temperature $T \approx 0°K$ would have been zero but owing to the existence of the Fermi energy ε_F, this internal energy density which is proportional to ε_F, is non zero.[21]

Interestingly it has been shown that at very high temperatures, a similar argument can lead to an exactly opposite effect viz., the Fermionization of Bosons.[22]

3. ONE AND TWO DIMENSIONAL BEHAVIOUR

We very briefly comment on what happens in the two and one dimensional cases in the context of the QMBH considerations seen above. These are two extreme idealizations because it is spin half that leads to and is responsible for three dimensions.[23] Side stepping this issue for the moment and also the fact that this corresponds to constrained Quantum systems, we observe that from the QMBH point of view this is opposite to the previous situation. We are in the high energy relativistic domain in the sense that the shrinkage of even a single dimension implies that we are already inside the Compton wavelength, and the concept of the particle inertial mass and other properties become questionable (cf. refs. 1 and 2).

In this case, we encounter mostly the negative energy components which exhibit the lefthanded behaviour: In the case of the neutrino also there is a similar situation but this is due to the fact that a Fermion without mass has infinite, or in practise very large Compton wavelength so that we are in the negative energy component region (cf. ref. 2). We can now argue, exactly as we did in the previous section for the QFT Hamiltonian (6), that the question of transition to empty states of the Dirac sea of opposite sign of energy does not arise as these states are unavailable. Whence we can use commutators instead of anti commutators. These conclusions can be easily verified.

Indeed in two and one dimensions the relativistically covariant equations have two components and exhibit handedness.[24] Infact as is well known, to build a Lagrangian with invariant mass, we need four components, that is we get back to three dimensional space.

To further clarify this situation and demonstrate self consistency within the QMBH model let us take the Lorentz covariant equation in one (spatial) dimension, in a well known and obvious notation:

$$\left(\imath^{\mu}\gamma^{\mu}\,\partial_{\mu} - \frac{mc}{\hbar}\right)\psi = 0$$

Apart from the fact that we get the left handed solution, it must be noticed that the mass term (or the energy operator term) is not accompanied by the usual factor,

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which in fact gives the positive and negative energy solutions and the Zitterbewegung (cf. ref. 6 and 13), and which leads to-equations like (1) and (2), or to the QMBH bounded by the Compton wavelength, and inertial mass itself.[25]

Another way of looking at this is that if we work only with solutions of one sign, the current, or equivalently, the expectation value of the velocity operator $c\vec{\alpha}$, is given by (cf. ref. 13),

$$J^{+} = \langle c\vec{\alpha} \rangle_{+} = \langle c^2 p/E \rangle_{+} = \langle v_{gp} \rangle_{+}$$

which is a contradiction, because, $c\vec{\alpha}$ has eigen values $\pm c$, whereas we require $\langle v_{gp} \rangle < c$, if the particle has mass. So, either the particle has no invariant mass, or both positive and negative energy solutions have to be included. In our case, we have neutrino like particles.

Indeed in low dimensions we have Fermion–Bosonic Transformation and other statistics like anyon statistics.[15, 16] We can infact show that the assembly behaves as if it is at a temperature below the Fermi Temperature: The average energy per unit length in one dimension is given by

$$e = \frac{\pi (kT)^2}{6\hbar v_F} \tag{7}$$

where $v_F \equiv \hbar\pi(N/L)/m$, L being the length of the one dimensional wire and N the number of Fermions therein. This is the one dimensional version of the Stephan Boltzmann law for radiation.[26] Denoting the average interparticle distance,

$$\frac{L}{N} \equiv (v)^{1/3}$$

and using the fact that[14]

$$kT_F = \left(\frac{\hbar^2}{2m}\right)\left(\frac{6\pi^2}{v}\right)^{2/3}$$

and remembering that,

$$kT = ev^{1/3}$$

we can easily deduce from (4) that,

$$T = \tfrac{3}{5} T_F$$

Interestingly this not only shows that the temperature is below the Fermi temperature, but also that the gas is in the ground state,[14] whatever be the temperature.

4. DISCUSSION AND CONCLUSION

Fermions and Bosons are divided into two different compartments, obeying Fermi–Dirac and Bose–Einstein statistics respectively. While this is true in general, there are special situations, for example at very low temperatures or in low dimensions where the distinction gets some what blurred leading to Bosonization or Semionic effects. The QMBH model predicts such a Bosonization effect for Fermions, at energies corresponding to scales much larger than the Compton wavelength. Indeed such an anomalous behaviour is found experimentally in the superfluidity of He^3: Though this is sought to be explained in terms of the conventional BCS theory, the fact is that there are inexplicable anomalous features (cf. ref. 12).

The model also predicts handedness and the blurring of Fermi–Dirac statistics in the two and one dimensional cases, which indeed are known features, and are confirmed by recent experiments.

Finally it may be mentioned that very recent experimental results on carbon nanotubes[27–30] exhibit the one dimensional nature of conduction and behaviour like low temperature quantum wires thus confirming the results discussed.

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